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The Economics of Truncation Error Control Versus High Resolution

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1. Derivations

Consider the linear equation

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = 0$$

and its centered-difference analog

$$v_{2t} + U\,\bar{v}_{2x} = 0 \qquad (1)$$

where

$$v_{2t} \equiv (v_{j,n+1} - v_{j,n-1})\,/2\Delta t$$

$$v_{2x} \equiv (v_{j+1,n} - v_{j-1,n})\,/2\Delta$$

j and n are serial numbers of grid points in x and t, respectively; Δt is the time step; Δ is distance between adjacent grid points. The superposed bar $(\bar{\ })$ in (1) represents an as yet unspecified discrete linear operation, such as smoothing.

I will regard the solutions of (1) as sums of components

$$v_{\mu,\nu} = A_{\mu,\nu} \cos\,(\mu j - \nu n) \qquad (2)$$

Substitution from (2) into (1) yields the frequency equation

$$\sin \nu = \frac{U\Delta t}{\Delta}\, w_{\mu} \sin \mu \qquad (3)$$

The function, w, of μ in (3) is the response of the bar-operator in (1).

For bounded solutions,

$$\frac{U\Delta t}{\Delta}\, w_{\mu}\ \sin \mu < 1$$

for otherwise

$$\sin \nu = \cosh\,(\nu - \frac{\pi}{2})$$

and the components (2) would contain exponentials in n.

Now, when the bar-operator is nul, w = 1, and a sufficient condition for stability is the familiar

$$\frac{U\Delta t}{\Delta} < 1$$

because the maximum absolute value of $\sin \mu$ is 1. If generally, $w \neq 1$, then $U\Delta t/\Delta$ must be adjusted to the maximum absolute value of $w \sin \mu$.

I will define the response of the difference-estimate in (1) as

$$w'_{\mu} = \frac{\overline{(v_{\mu,\nu})_{2x}}}{\frac{\partial v_{\mu,\nu}}{\partial x}} \qquad (4)$$

Call w' in the case of the nul bar-operator (w = 1), w°.

By substitution from (2) into (4) we find

$$w^{\circ}_{\mu} \equiv \frac{(v_{\mu,\nu})_{2x}}{\frac{\partial v_{\mu,\nu}}{\partial x}} = \frac{\sin \mu}{\mu} \qquad (5)$$

A useful characteristic of the function w°_{μ} is that it is very nearly a linear function of $(\cos \mu)$ in the range $0 < \mu < \frac{1}{2}\pi$. Compared to the function

$$w''_{\mu} = \tfrac{1}{3}(2 + \cos \mu)$$

they both are unity at $\mu = 0$, and have derivatives with respect to $(\cos \mu)$ of 1/3 there. At $\mu = \frac{1}{2}\pi$, $w^{\circ}_{\mu} = 2/\pi$, and $w''_{\mu} = 2/3$, nearly the same.

Fourth-order truncation error control amounts to using the bar-operator whose response is

$$w = \tfrac{1}{3}(4 - \cos \mu). \qquad (6)$$

The maximum of $(w \sin \mu)$, with w defined as in (6), is 1.37222 and occurs at $\mu = .57215\,\pi$ $(\cos \mu = -0.22474)$. With fourth-order truncation error control, therefore, the time-step must be reduced by 27.1% $(1 - 1.37222^{-1})$, increasing the number of time-steps in a forecast by 37.2%.

There is an additional penalty, in the number of calculations needed to apply the bar-operator. This is minimal, however, since the bar-operator, whose response is (6), has the simple weighting pattern:

$$-\frac{1}{6} \quad \frac{4}{3} \quad -\frac{1}{6}$$

On the other hand, consider a reduction in grid-size by a factor, α, retaining second-order differencing ($w = 1$). For stability, the time-step must also be reduced by the factor, α. In a numerical weather model with two space dimensions, the number of calculations will therefore increase by the factor α^{-3}.

Now, for a given wave-length, let's ask what α must be to achieve the same correction that fourth-order differences give. Since j', the serial number in the new more highly resolved grid, is related to j by

$$\alpha j' = j$$

we may write for the components in the new grid

$$v_{\mu', \nu'} = A'_{\mu', \nu'} \cos\left(\frac{\mu'}{\alpha} j - \nu' n'\right)$$

and, for the same wave-length,

$$\mu = \frac{\mu'}{\alpha}$$

The response for the first centered-difference is

$$w^{\circ}_{\mu'} = \frac{\sin \mu'}{\mu'} = \frac{\sin \mu\alpha}{\mu\alpha}$$

The problem then becomes that of solving

$$\sin \mu\alpha = \tfrac{1}{3}\alpha(4 - \cos \mu) \sin \mu$$

for α, given μ. This was done on one of the new small hand-held programmable calculators.

2. Results

The table below summarizes the results for various μ's. In the table, N is the wave-length in units of the larger grid-increment:

$$N = 2\pi/\mu$$

N	w°	w'	α	α^{-3}
15	.971	.9990	.185	157
12	.955	.9976	.231	81.7
10	.936	.9950	.275	48.3
8	.900	.988	.339	25.6
6	.827	.965	.441	11.7
5.198	.774	.940	.5	8
4	.637	.849	.621	4.18
3	.414	.620	.769	2.20
2.4	.191	.310	.891	1.41
2	0	0	1	1

The first entry, for 15Δ-long waves, is not very interesting. Although 157 times as much calculation would be required, with higher resolution, to reach the truncation error level of fourth-order differences, the uncorrected lower resolution has an error of less than 3%. Furthermore, it would be unreasonable to require this to be reduced to 0.1%.

The entry for the 8Δ-long wave is more interesting. With the grid-increments of 200 to 400 km in use in NMC operational models, these components are 1600-3200 km long -- below the storm scale, but important contributors to shapes of storms. Here, fourth-order differences reduce error from 10% to 1.2%, which if achieved by increased resolution alone, would increase the number of calculations over 25 times.

The entry for 6Δ-long waves, which have a large (17.3%) error without correction, shows that their corrected error (3.5%) is less than the uncorrected error (4.5%) for 12Δ-long waves. To achieve the same result with higher resolution alone would increase the calculations 11.7 times.

3. Conclusions

The conclusion is that, although fourth-order differencing techniques have difficulties not discussed here, they are extremely interesting vis-a-vis higher resolution from the standpoint of economy. They lead to such small errors down to wave-lengths 6Δ long or so, that they are also interesting compared with orthogonal-function techniques.

Even if semi-implicit techniques were combined with higher resolution, the fourth-order techniques would sustain their relative economy. Semi-implicit methods yield only a factor of 4 in reduction of numbers of calculations, without affecting the truncation error discussed here. Our results as shown in the table in the previous section indicate considerably more benefit than that for waves longer than 6Δ. There is, of course, no reason to suppose that semi-implicit methods and fourth-order differences are mutually exclusive.